Exercise 5 Hints for: Feature Extraction and Uncertainty Propagation

1 General

- to output matlab variables write the variable's name without trailing ;.
  `>> a = [1; 2; 5; 9];`
  `>> a [1; 2; 5; 9]`
- arrays in matlab start with index 1, instead of 0 (as most of the programming languages).
  `>> a(2)`
  `2`
- to execute a script type its name without the trailing *".m"* extension.
- diag([1 2 3]) creates a diagonal 3 x 3 matrix, with the vector as diagonal elements.
- diff() returns a vector of differences between elements.
  `>> diff(a)`
  `[1; 3; 4]`

2 Math

- the questions 3 a, b, c require math and not matlab. You can work on them in parallel.
- *ground truth* is a technical term, describing the error-free real world reference for an estimation. In general this is hard to obtain.
- the Jacobian of a matrix f is defined as:

$$\left[\nabla_Y f^T(Y)\right]^T = f(Y)\nabla_Y^T = f(Y)\left[\frac{\delta}{\delta y_1}\cdots\frac{\delta}{\delta y_i}\cdots\frac{\delta}{\delta y_N}\right] \tag{1}$$

- the inverse of a $2x2$ matrix is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

3 Line Estimation

Given N points (y_i, x_i), which supposedly belong to the same line with (b_0, b_1) for ordinate and tangens of the angle. The difference to the line is expressed as d_i. We can write:

$$\begin{aligned} \tilde{y}_1 &= b_0 + b_1 x_1 + d_1 \\ &\vdots \\ \tilde{y}_i &= b_0 + b_1 x_i + d_i \\ &\vdots \\ \tilde{y}_N &= b_0 + b_1 x_n + d_N \end{aligned} \tag{3}$$

Or as a matrix:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \tilde{y}_1 - d_1 \\ \vdots \\ \tilde{y}_i - d_i \\ \vdots \\ \tilde{y}_N - d_N \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ & \vdots \\ x_i & 1 \\ & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \tag{4}$$

$$Y = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_i \\ \vdots \\ \tilde{y}_N \end{bmatrix} \approx \begin{bmatrix} \tilde{y}_1 - d_1 \\ \vdots \\ \tilde{y}_i - d_i \\ \vdots \\ \tilde{y}_N - d_N \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 & 1 \\ & \vdots \\ x_i & 1 \\ & \vdots \\ x_N & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix} \tag{5}$$

Solving for B:

$$XB = Y \tag{6}$$

$$X^T XB = X^T Y \tag{7}$$

$$B = (X^T X)^{-1} X^T Y \tag{8}$$

$$X^T X = \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & N \end{bmatrix} \tag{9}$$

$$X^T Y = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix} \tag{10}$$

$$B = \frac{1}{n \sum_i x_i^2 - (\sum_i x_i)^2} \begin{bmatrix} \sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i \\ n \sum x_i y_i - \sum x_i \sum y_i \end{bmatrix} \tag{11}$$

For the covariance of B, we start with the covariance of the points C_Y:

$$C_Y = diag(\sigma_{y1}^2, \ldots, \sigma_{yi}^2, \ldots, \sigma_{yN}^2) \tag{12}$$

$$B_Y = \left[\nabla_Y B^T(Y)\right]^T = \left[\nabla_Y B^T\right]^T \tag{13}$$

$$= B(Y) \left[\frac{\delta}{\delta y_1}, \ldots, \frac{\delta}{\delta y_i}, \ldots, \frac{\delta}{\delta y_N}\right] \tag{14}$$

$$C_B = B_Y C_Y B_Y^T \tag{15}$$